**W49**. The sequence $(x_n)_{n\geq 0}$ is defined by $x_0 = 1$ and

$$x_{n+1} = \frac{\sqrt{3}\,x_n - 1}{x_n + \sqrt{3}},\ n = 0, 1, 2, \ldots$$

Find $x_{2021}$.

**Ovidiu Bagdasar.**

**Solution by Arkady Alt, San Jose, California, USA.**

Let $(\varphi_n)_{n\geq 0}$ be sequence defined by $\varphi_0 = \frac{\pi}{4}$ and $\varphi_{n+1} = \varphi_n + \frac{\pi}{6}, n \in \mathbb{N} \cup \{0\}$ that is $\varphi_n = \frac{\pi}{4} + \frac{n\pi}{6}, n \in \mathbb{N} \cup \{0\}$.

Since $x_0 = \cot\varphi_0$ and for any $n \in \mathbb{N} \cup \{0\}$ assuming $x_n = \cot\varphi_n$ we obtain $x_{n+1} = \frac{\sqrt{3}\,x_n - 1}{x_n + \sqrt{3}} = x_{n+1} = \frac{\sqrt{3}\,\cot\varphi_n - 1}{\cot\varphi_n + \sqrt{3}} = \cot\left(\varphi_n + \frac{\pi}{6}\right) = \cot\varphi_{n+1}$ then by Math Induction $x_n = \cot\varphi_n = \cot\left(\frac{\pi}{4} + \frac{n\pi}{6}\right), n \in \mathbb{N} \cup \{0\}$.

In particular, $x_{2021} = \cot\left(\frac{\pi}{4} + \frac{2021\pi}{6}\right) = \cot\left(\frac{\pi}{4} + \frac{5\pi}{6}\right) = \frac{1 \cdot \cot\frac{5\pi}{6} - 1}{\cot\frac{5\pi}{6} + 1} =$

$$\frac{1 \cdot \left(-\sqrt{3}\right) - 1}{\left(-\sqrt{3}\right) + 1} = 2 + \sqrt{3}.$$